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HERSCOVICI's CONJECTURE ON PRODUCTS OF STARS, FAN GRAPHS

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Abstract.
Given a connected graph G,distribute k pebbles on its vertices in some configuration C.Specifically a configuration on a graph G is a function f from V(G) to $N \cup \{0\}$ representing an assignment of pebbles on G.We call the total number of pebbles,k, the size of the configuration.A pebbling move is defined as the removal of two pebbles from a vertex and addition of one of those pebbles on an adjacent vertex.The pebbling number of a connected graph G is the smallest number f(G) such that,however f(G) pebbles are distributed on the vertices of G,we can move a pebble to any vertex by a sequence of pebbling moves.The t-pebbling number $f_t(G)$ of a simple connected graph G is the smallest positive integer such that for every distribution of $f_t(G)$ pebbles on the vertices of G,we can move t pebbles to any target vertex by a sequence of pebbling moves.Graham conjectured that For any connected graphs G and H ,$f(G\times H) \leq f(G)f(H)$.Herscovici further conjectured that $f_{st}(G\times H) \leq f_s(G)f_t(H)$ for any positive integers s and t.In this paper we show that Herscovici's conjecture is true when G is a star,fan graphs and H is a graph satisfying the 2t- pebbling property.

Keywords. star,fan graphs graph,t-pebbling number,Herscovici's conjecture.

1.1. **Introduction**

The pebbling number is known for many simple graphs including paths, cycle and trees, but is unknown for most graphs and is hard to compute for any given graph that does not fall into one of these classes. Therefore, it is an interesting question if there is an information we can gain about the pebbling number of more complex graphs from the knowledge of the pebbling number of some graphs for which we know. In the first paper on graph pebbling [1] Chung proposed the following conjecture. The conjecture is perhaps the most compelling open question in graph pebbling known as **Grahams conjecture.**

Conjecture1.1.1(Graham ([7]). For all graphs G_1 and G_2, we have $f(G_1 \times G_2) \leq f(G_1)f(G_2)$. which is $f(Q_d)=2^d$. The hypercube is formed by a product of length two paths; $Q_d=Q_{d-1} \times P_2$. And we know that $2^d=f(Q_d)=f(Q_{d-1})f(P_2)=2^{d-1}2$. In addition to this the result has been shown to be true for the product of trees, the product of some specific cycles, and the product of a complete graph and any graph with the two pebbling property. In proving Graham's conjecture on graph pebbling two properties are used in the literature. They are the 2-pebbling property[7] and the odd 2-pebbling property**.**There are a number of results that support Graham's conjecture, the first of pebbling property. In [3], Lourdusamy has defined the 2t-pebbling property of a graph.

Definition 1.1.2 ([3]). Given t-pebbling number of G, let p be the number of pebbles of G, let q be the number of vertices with at least one pebble. We say that G satisfies the 2t-pebbling property if it is possible to move 2t pebbles to any specified target vertex of G starting from every configuration in which $p \geq 2f_t(G)-q+1$ or equivalently $p+q>2f_t(G)$ for all t.

The direct product of two graphs is defined as follows:

Definition 1.1.3 [2]. If $G=(V_G,E_G)$ and $H=(V_H,E_H)$ be two graphs, the direct product of G and H is the graph G×H whose vertex set is the cartesian product $V_{G \times H}=V_G \times V_H=\{(x,y);x \in V_G, y \in V_H)$ and whose edges are given by $E_{G \times H}=\{(x,y),(x',y');x=x' \text{ and } (y,y') \in E_H \text{ or } (x,x') \in E_G \text{ and } y=y'\}$

We find the following theorems in [5] and [6]

Theorem 1.1.4 [5]. Let $K_{1,n}$ be on n –star where n>1. Then $f_t(K_1,n)=4t+n2$

Theorem 1.1.5 [6]. Let F_n be a fan graph on n vertices in order. For $n\geq 4$, $f_t(F_n)=4t+n-4$.

With regard to the 2t-pebbling property we find the following results in [3], [4],[5] ,[6]and [7].

Theorem 1.1.6 [7] All diameter two graphs satisfy the two pebbling property.

Theorem 1.1.7 [3] All paths satisfy the 2t-pebbling property for all t.

Theorem 1.1.8 [4] Let K_n be a complete graph on n vertices. Then K_n satisfies the 2t-pebbling property for all t.

Theorem 1.1.9 [5] The star graph $K_{1,n}$ where n>1 satisfies the 2t-pebbling property.

Theorem 1.1.10 [6] Fan graphs satisfy the 2t-pebbling property.

With regard to the t-pebbling conjecture on products of graphs we find the following Theorems in [4], and [6].

Theorem 1.1.11 [4] Let $(K_{1,n})(n>1)$ be a star. If G satisfies the 2t-pebbling property, then $f_t(K_{1,n} xG)\leq f(K_{1,n})f_t(G)$ for all t.

Theorem 1.1.12 [6] Let F_n be a fan graph on n vertices $v_o,v_1,\ldots.v_{n-1}$ in order. If G satisfies the 2t-pebbling property, then $f_t(F_n,xG)\leq f(F_n)f_t(G)$ for all t.

Lourdusamy conjectured as

Conjecture 1.1.13 (Lourdusamy[3]). For any connected graphs G and H we have $f_t(G\times H)\leq f(G)f_t(H)$ for all t.

This conjecture is called the t –pebbling conjecture and Lourdusamy proved it when G is an even cycle and H satisfies a variation of the two-pebbling property. Herscovici conjectured as

Conjecture1.1.14. For any connected graphs G and H, We have $f_{st}(G\times H)\leq f_s(G)\, f_t(H)$ for all s,t.

In this paper, we prove that Herscovici's conjecture is true when G is a star,fan graphs and H is a graph having the 2t pebbling property.

Theorem 1.1.15 [3]. The t-pebbling number of the path on n vertices is given by $f_t(P_n)=t2^{n-1}$.

Theorem 1.1.16 [3]. Let $K_{1,n}$ be on n –star where n>1. Then $f_t(K_1,n)=4t+n2$

Theorem 1.1.17 [6]. Let F_n be a fan graph on n vertices in order. For $n\geq 4$, $f_t(F_n)=4t+n-4$.

With regard to the 2t-pebbling property we find the following results in [2] , [3], [4], [5]and [6].

Theorem 1.1.18 [5] All diameter two graphs satisfy the two pebbling property.

Theorem 1.1.19 [2] All paths satisfy the 2t-pebbling property for all t.

Theorem 1.1.20 [3] Let K_n be a complete graph on n vertices. Then K_n satisfies the 2t-pebbling property for all t.

Theorem 1.1.21 [4] The star graph $K_{1,n}$ where n>1 satisfies the 2t-pebbling property.

Theorem 1.1.22 [6] Fan graphs satisfy the 2t-pebbling property.

With regard to the t-pebbling conjecture on products of graphs we find the following Theorems in [4] .

Theorem 1.1.23 [4] Let $(K_{1,n})(n>1)$ be a star. If G satisfies the 2t-pebbling property, then $f_t(K_{1,n}\, xG)\leq f(K_{1,n})f_t(G)$ for all t.

Theorem 1.1.24 [4] Let F_n be a fan graph on n vertices $v_o,v_1,\ldots.v_{n-1}$ in order. If G satisfies the 2t-pebbling property, then $f_t(F_n,xG)\leq f(F_n)f_t(G)$ for all t.

1.2. Herscovici's conjecture on products of stars

Theorem 1.2.1 Let $K_{1,m}$ (m>1) be a star. If G satisfies the 2t-pebbling property. They $f_{st}(K_{1,m}xG)\leq f_s(K_{1,m})f_t(G)$ for all s,t.

Proof : Let $V(K_{1,m})=U\bigcup W$ where U={u}and $W=\{w_1,w_2,\ldots.w_m\}$.We use induction on s to prove that $f_{st}(K_{1,m}xG)\leq f_s(K_{1,m})f_t(G)$ for all s,t.

For s=1, theorem is true by Theorem 1.1.23.

We take s>1. Then there are at least $(m+6)f_t(G)$ pebbles on the graph. Let a be the number of pebbles on {u} xG and a_i be the number of pebbles on $\{w_i\}$xG. Let $y \in G$.

Case 1. Suppose the target vertex is (u,y). By pigeonhole principle, $a_i \geq 2f_t(G)$ for some i=1,2,…m. Then from $\{w_i\}$xG, $f_t(G)$ pebbles can be moved to {u}xG and hence t pebbles can be moved to (u,y). This leaves us with at least (4s+n-4) $f_t(G)$ pebbles. We can place (s-1) t additional pebbles on (u,y) by induction.

Case 2. Suppose the target vertex is $(w_i,y)$for some i=1,2,……m. We claim that either {u}xG has at least $2f_t(G)$ pebbles or there exists one copy in $\{W-w_i\}$xG with at least $4f_t(G)$ pebbles or there are at least two copies in $\{W-w_i\}$xG with at least $2f_t(G)$ pebbles each. Otherwise the total number of pebbles placed will be at most $(m+2)f_t(G)$.

Case 2.1: If {u}xG has at least $2f_t(G)$ pebbles then we can put $f_t(G)$ pebbles on $\{w_i\}$xG for some i. Then t pebbles can be moved to (w_i,y). With the remaining (4(s-1)+m) $f_t(G)$ pebbles we can move (s-1) t additional pebbles on (w_i,y).

Case 2.2 If there exists a copy in $(W-\{w_i\})$xG with at least $4f_t(G)$ or there are at least two copies in $(W-\{w_i\})$xG with at least $2f_t(G)$ pebbles then we can move $f_t(G)$ pebbles to {u}xG using at most $4f_t(G)$ pebbles. Hence t pebbles can be moved to (u,y). This leaves us with at least $(4(s-1)+(m-2))f_t(G)$ pebbles which would suffice to put (s-1) t additional pebbles on (w_i,y) by induction. □

Corollary 1.2.2. Let $K_{1,m}(m>1)$ be a star and P_n be a path on n vertices. Then $f_{st}(K_{1,m}xP_n) \leq f_s(K_{1,m})f_t(P_n)$ for all s,t.

proof : The corollary follows from Theorem 1.2.1 and Theorem 1.1.19 □

Corollary 1.2.3. Let $K_{1,m}(m>1)$ be a star and F_n be a fan graph on n vertices. Then $f_{st}(K_{1,m}xF_n)<f_s(K_{1,m})f_t(F_n)$ for all s,t.

Proof : The corollary follows form Theorem 1.2.1 and Theorem 1.1.22 □

Corollary 1.2.4. Let $K_{1,m}(m>1)$ be a star and K_n be a complete graph. Then $f_{st}(K_{1,m}xK_n) \leq f_s(K_{1,m})f_t(K_n)$ for all s,t.

Proof : The corollary follows from Theorem 1.2.1 and Theorem 1.1.20 □

Corollary 1.2.5. Let $K_{1,m}(m>1)$ be a star. Then $f_{st}(K_{1,m}xK_{1,n})\leq f_s(K_{1,m})f_t(K_{1,n})$ for all s,t.

Proof : The corollary follows from Theorem 1.2.1 and Theorem 1.1.21 □

We have proved that conjecture 1.1.14 is true for all the products of a star by a (i) Path (ii) Fan (iii) Complete graph (iv) Star

1.3 Herscovici's conjecture on products of fan graphs

Theorem 1.3.1. Let F_n be a fan graph on n vertices $v_o,v_1,v_2,\ldots v_{n-1}$ in order. If G satisfies the 2t-pebbling property then $f_{st}(F_nxG)\leq f_s(F_n)f_t(G)$ for all s,t., $n\geq 3$.

Proof: We take the n copies of G i.e. $\{v_o\}xG$, $\{v_1\}xG,\ldots\{v_{n-1}\}xG$ respectively as $G_0,G_1,\ldots,G_{n-1}$ and let a_i be the number of pebbles on G_i with p_i occupied vertices where i=0,1,2,…n-1. Let $y\in G$.

Without loss of generality we assume that $(v_0,y)$is the target vertex. (v_0,y) is adjacent with each of (v_i,y) and $(v_{i-1},y)$where i=2,3,…,n-1. [If the target vertex is (v_i,y), then it is adjacent with each of (v_0,y) and (v_{i-1},y)]

We will prove the theorem by induction on s. The theorem is true by Theorem1.1.24, when s=1. Assume s>1. If $a_0>f_t(G)$, then we put t pebbles on (v_o,y). This leaves us with at least $(4s+n-5)f_t(G)$ pebbles which would suffice to put (s-1)t additional pebbles on (v_0,y). Hence assume $a_0\leq f_t(G)-1$.

If $a_{i-1}+a_0+a_i\geq 3f_t(G)$ pebbles, then $f_t(G)$ pebbles can be moved to G_0 and hence t pebbles can be moved to (v_0,y), then we are done. With the remaining $(4s+n-7)f_t(G)$pebbles, we can put (s-1)t additional pebbles on $(v_o,y)$by induction. Assume $a_{i-1}+a_o+a_i\leq 3f_t(G)-1$. Then the number of pebbles on the sub graph $(F_n-\{v_0,v_i,v_{i-1}\})xG$ will be at least $(4s+n-7)\ f_t(G)+1$ pebbles.

Hence $(F_n-\{v_0,v_i,v_{i-1}\})xG$ contains at least $(n+1)f_t(G)+1$ pebbles as s>1. By pigeonhole principle at least one of G_j(j=1,2,…n-1 where $j\neq$ i, i-1) receives at least $2f_t(G)$ pebbles hence $f_t(G)$ pebbles can be moved to G_0. Hence t pebbles can be moved to (v_0,y). This leaves us with at least $(4s+n-6)\ f_t(G)$ pebbles which would suffice to put(s-1) t addional pebbles on (v_0,y) by induction. □

Corollary 1.3.2 Let F_n be a fan graph on n vertices and P_m be a path on m vertices. Then $f_{st}(F_nxP_m)\leq f_s(F_n)f_t(P_m)$ for all s,t.

Proof : The corollary follows from Theorem 1.3.1 and Theorem 1.1.19 □

Corollary 1.3.3. Let F_n be a fan graph on n vertices and K_m be a complete graph on m vertices. Then $f_{st}(F_n x K_m) \leq f_s(F_n) f_t(K_m)$ for all s,t.

Proof : The corollary follows from Theorem 1.3.1 and Theorem 1.1.20 □

Corollary 1.3.4. Let F_n be a fan graph on n vertices and $K_{1,m}$ be an m-star (m>1). Then $f_{st}(F_n x K_{1,m}) \leq f_s(F_n) f_t(K_{1,m})$ for all s,t.

Proof : The corollary follows from theorem 1.3.1 and Theorem 1.1.21. □

Corollary1.3.5. Let F_n be a fan graph on n vertices Then $f_{st}(F_m x F_n) \leq f_s(F_m) f_t(F_n)$ for all s,t.

Proof : The corollary follows from Theorem 1.3.1 and Theorem 1.1.22. □

Thus we have proved that conjecture 1.1.14 is true for all the products of a complete graph by a (i)Path (ii) Fan (iii) Complete graph (iv) Star.

Is Herscovici's conjecture is true when G is a complete bipartite graph?

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